

Dissipative Cylindrical Collapse of Charged Anisotropic Fluid

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We have studied the dynamics of a cylindrical column of anisotropic, charged fluid which is experiencing dissipation in the form of heat flow, free-streaming radiation, and shearing viscosity, undergoing gravitational collapse. We calculate the Einstein-Maxwell field equations and, using the Darmois junction conditions, match the interior non-static cylindrically symmetric space-time with the exterior anisotropic, charged, cylindrically symmetric space-time. The behavior of the density, pressure and luminosity of the collapsing matter has been analyzed. From the dynamical equations, the effect of charge and dissipative quantities over the cylindrical collapse are studied. Finally, we have presented the solutions for this dissipative, cylindrical gravitational collapse and have discussed the significance from a physical standpoint.

I. INTRODUCTION

An important task in relativistic gravity and astrophysics is to build realistic models to describe the final fate of a star which has exhausted its nuclear fuel [1, 2]. While the star was still burning, the material was held up by the radiation pressure generated inside its core [3]. Once the fuel is exhausted, the star begins its collapse under the internal pull of gravity. The fate of any gravitational collapse depends upon the mass of the collapsing star [4]. A star with a mass of the order of the Sun will eventually collapse into a white dwarf, where the material is held up by the electron degeneracy pressure generated by electrons moving within it at high speed. A star which has about four to five times the solar mass will however settle into the state of a neutron star, in which even the atoms have collapsed under the enormous pressure of gravity, leading to the formation of minute electrically neutral regions inside it, of dimensions greater than the size of an atom [5]. This object made up purely of neutrons will be held up due to the neutron degeneracy pressure and will be about twenty kilometers wide. Even more massive stars will neither settle into a white dwarf nor a neutron star and the gravitational pull will be sufficient to collapse it into a black hole. These are one of the most exotic objects in the universe as they collapse to within the "Schwarzschild radius" and therefore no information is able to escape from within, what is called, the event horizon.

There has been an extensive study of collapse of dust and fluid under gravity since the formulation of General Relativity by Einstein. Contributions by S. Chandrasekhar, who worked out the maximum mass of a stable white dwarf star, known as the Chandrasekhar limit, to the collapse study of adiabatically flowing dust by Oppenheimer and Snyder [6], set the standard for research in this area. P. C. Vaidya [7] studied the external gravitational field of a stellar body giving out radiations. Misner and Sharp [9, 10] studied spherically symmetric collapse and contributed by giving an expression for the total energy. Others [11] studied the geometry of the Vaidya metric. K. Lake, L. Herrera, N.O. Santos and many others [12–21] have studied different cases of spherically symmetric fluids undergoing collapse. A realistic model of such a collapse should include a radial heat flux as the process of gravitational collapse is highly dissipative [22, 23]. Recently, Sharma and Tikekar [24] has studied the evolution of non-adiabatic collapse of a shear-free spherically symmetric stellar configuration with anisotropic stresses accompanied with radial heat flux. The effects of electromagnetic field on the energy density inhomogeneity in the relativistic self-gravitating fluids for spherically symmetric spacetime has been studied by Sharif and Bashir [25]. There has also been extensive studies on quasi-spherical and self-similar collapse [26] over the years. An extensive list of papers on the study of gravitational collapse can be found in [1] and [24].

The study of non-spherical gravitational collapse has gained in momentum following the discovery of cylindrical and plane gravitational waves. Bronnikov and Kovalchuk [27] were the pioneers in the investigations on non-spherical gravitational collapse. Ashtekar and Varadarajan obtained the expression for the energy per-unit length (along the symmetry direction) of gravitational waves with a spacelike symmetry in 3+1 dimensions, with cylindrical waves as a special case [28]. The definition of cylindrical symmetry in general relativity as a specialization of the case of axial symmetry has been considered by Carot *et al.* [29] and Barnes [30]. Chiba [31] studied the case of cylindrical dust collapse. Others [32–34] investigated various aspects of cylindrical collapse of counter rotating dust and rotating cylindrical shells. Hayward studied gravitational waves, black holes and cosmic strings in cylindrical symmetry. Considering the most general vacuum cylindrical spacetimes, Goncalves [35] has demonstrated that there are no trapped cylinders in the spacetime, and has presented a formal derivation of Thorne's C-energy, based on a Hamiltonian

reduction approach. Di Prisco *et al.* [37] studied shear-free cylindrical gravitational collapse for an interior non-rotating fluid with anisotropic pressures and exterior vacuum Einstein-Rosen spacetime. The case of the collapse of a heat conducting charged anisotropic fluid cylinder have been studied by Sharif and Abbas [38] and that of a charged fluid cylinder with shear viscosity by Sharif and Fatima [39].

In this paper, we have extended the work done by Sharif and Abbas [38] and Sharif and Fatima [39] and have examined the effect of charge, heat flow, radiation and shear viscosity on the gravitational collapse of a cylindrical column of anisotropic fluid. We have formulated the field equations for the source and have employed the Darmois junction condition for the smooth matching of the interior non-static cylindrically symmetric space-time and the exterior anisotropic, viscous, charged, cylindrically symmetric space-time at the boundary Σ , to derive a few results. The paper is organized as follows: In the next section, we describe the gravitational source along with the corresponding physical parameters like the expansion scalar, acceleration, shear tensor and the Einstein-Maxwell field equations. In section III, the exterior metric and the junction conditions are given. The results, including the dynamical equations and the solutions are presented in section IV. The summary of this whole exercise is given in section V.

II. THE INTERIOR METRIC AND THE FIELD EQUATIONS

We consider a collapsing cylinder filled with an anisotropic, charged fluid and undergoing dissipation in the form of heat flow, free-streaming radiation, and shearing viscosity, bounded by a timelike cylindrical three-surface Σ , which divides the space-time into two distinct 4-dimensional manifolds V^+ and V^- .

A. The interior spacetime

For the interior V^- space-time we take the general non-static cylindrically symmetric metric in the comoving coordinates given by

$$ds_-^2 = -A^2(t, r)dt^2 + B^2(t, r)dr^2 + C^2(t, r)(d\theta^2 + dz^2) \quad (1)$$

In order to represent cylindrical symmetry, the range of coordinates is required to be as follows:

$$\begin{aligned} -\infty < t < \infty, \quad 0 \leq r < \infty, \\ 0 \leq \theta \leq 2\pi, \quad -\infty < z < +\infty, \end{aligned}$$

with the coordinate labels, $x^0 = t$, $x^1 = r$, $x^2 = \theta$ and $x^3 = z$. The interior energy-momentum tensor is given, according to relativistic hydrodynamics, as

$$T_{\alpha\beta} = (\mu + P_\perp)V_\alpha V_\beta + P_\perp g_{\alpha\beta} + (P_r - P_\perp)\chi_\alpha \chi_\beta + V_\alpha q_\beta + V_\beta q_\alpha + \epsilon l_\alpha l_\beta - 2\eta\sigma_{\alpha\beta}, \quad (2)$$

where, $\mu \rightarrow$ energy density, $P_\perp \rightarrow$ tangential pressure, $P_r \rightarrow$ radial pressure, $q^\alpha \rightarrow$ heat flux, $\epsilon \rightarrow$ the radiation density, $V^\alpha \rightarrow$ 4-velocity of the fluid, $\chi^\alpha \rightarrow$ unit 4-velocity in the radial direction, $l^\alpha \rightarrow$ a null 4-vector and $\eta \rightarrow$ coefficient of shearing viscosity > 0 respectively.

The shear tensor $\sigma_{\alpha\beta}$, the 4-acceleration a_α and the expansion Θ are defined as

$$\sigma_{\alpha\beta} = \frac{1}{2}((V_{\alpha;\beta} + V_{\beta;\alpha}) + (a_\alpha V_\beta + a_\beta V_\alpha)) - \frac{1}{3}\Theta(g_{\alpha\beta} + V_\alpha V_\beta), \quad (3)$$

$$a_\alpha = V_{\alpha;\beta}V^\beta, \quad (4)$$

and

$$\Theta = V_{;\alpha}^\alpha. \quad (5)$$

Since we have assumed comoving coordinates for the interior metric, we have

$$V^\alpha = A^{-1}\delta_0^\alpha, \quad \chi^\alpha = B^{-1}\delta_1^\alpha, \quad l^\alpha = A^{-1}\delta_0^\alpha + B^{-1}\delta_1^\alpha, \quad q^\alpha = B^{-1}q\delta_1^\alpha, \quad (6)$$

such that

$$V^\alpha V_\alpha = -1, \quad \chi^\alpha \chi_\alpha = 1, \quad \chi^\alpha V_\alpha = 0, \quad q^\alpha V_\alpha = 0, \quad l^\alpha V_\alpha = -1, \quad l^\alpha l_\alpha = 0. \quad (7)$$

In view of the equations (4), (5) and (6), we obtain the acceleration and the expansion scalar as follows:

$$a_\alpha = \frac{A'}{A} \delta_\alpha^1, \quad (8)$$

$$\Theta = \frac{1}{A} \left(\frac{\dot{B}}{B} + 2 \frac{\dot{C}}{C} \right). \quad (9)$$

Using (3) to (6), we obtain the non-zero components of the shear tensor as

$$\sigma_{11} = \frac{2}{\sqrt{3}} B^2 \sigma, \quad \sigma_{22} = \sigma_{33} = -\frac{1}{\sqrt{3}} C^2 \sigma. \quad (10)$$

The shear scalar σ defined by

$$\sigma^2 = \frac{1}{2} \sigma_{ij} \sigma^{ij} \quad (11)$$

is therefore obtained as

$$\sigma = \frac{1}{\sqrt{3}A} \left(\frac{\dot{B}}{B} - \frac{\dot{C}}{C} \right). \quad (12)$$

As defined by Chiba [31], a cylindrically symmetric spacetime may be defined locally by the existence of two commuting, spacelike, Killing vectors, such that the orthogonal space is integrable. For such a spacetime, there exist coordinates (θ, z) such that the Killing vectors are $(\xi_\theta, \xi_z) = (\partial/\partial\theta, \partial/\partial z)$. The existence of cylindrical symmetry about an axis implies that the orbits of one of these vectors are closed but those of the other is open. Each of these Killing vectors must be hypersurface orthogonal. The norms of these Killing vectors are invariants [40], namely the circumferential radius

$$\zeta = \sqrt{\xi_\theta \cdot \xi_\theta^b} = \sqrt{\xi_{(2)a} \xi_{(2)}^a},$$

and the specific length

$$\ell = \sqrt{\xi_z \cdot \xi_z^b} = \sqrt{\xi_{(3)a} \xi_{(3)}^a}$$

with $\xi_{(2)} = \partial_\theta$, $\xi_{(3)} = \partial_z$, under the sign convention that spatial metrics are positive definite, the dot representing contraction and the flat b represents the covariant dual with respect to the space-time metric. The gravitational energy per specific length in a cylindrically symmetric system (also known as C-energy) as defined by Thorne [41] and modified by him to render it finite in spacetime, is given by

$$E = \frac{1}{8} (1 - l^{-2} \nabla^a \tilde{r} \nabla_a \tilde{r}), \quad (13)$$

for which

$$\tilde{r} = \zeta l,$$

where the areal radius is \tilde{r} and E is the gravitational energy per unit specific length of the cylinder.

Analogous to the Misner and Sharp energy for spherical symmetry [42], the specific energy of the cylinder due to the electromagnetic field is therefore given by

$$E' = \frac{l}{8} + \frac{C}{2} \left(\frac{\dot{C}^2}{A^2} - \frac{C'^2}{B^2} \right) + \frac{s^2}{2C}. \quad (14)$$

B. Electromagnetic energy tensor and Maxwell's equations

The electromagnetic energy-momentum tensor for the charged fluid is given by

$$T_{\alpha\beta}^{(em)} = \frac{1}{4\pi} \left(F_\alpha^\gamma F_{\beta\gamma} - \frac{1}{4} F^{\gamma\delta} F_{\gamma\delta} g_{\alpha\beta} \right). \quad (15)$$

and the corresponding Maxwell's equations are

$$F_{\alpha\beta} = \psi_{\beta,\alpha} - \psi_{\alpha,\beta}, \quad (16)$$

$$F^{\alpha\beta}{}_{;\beta} = 4\pi J^\alpha, \quad (17)$$

where $F_{\alpha\beta}$ is the electromagnetic field tensor, ψ_α is the corresponding four potential and J_α is the four current density vector. Since the charge is comoving with the fluid, there is no magnetic field, and the four current density is proportional to the four velocity i.e. we have

$$\psi_\alpha = \psi \delta_\alpha^0, \quad J^\alpha = \rho V^\alpha, \quad (18)$$

where $\psi(t, r)$ is an arbitrary function and $\rho(t, r)$ is the charge density.

From the Maxwell's equations we obtain

$$\psi'' - \left(\frac{A'}{A} + \frac{B'}{B} - 2\frac{C'}{C} \right) \psi' = 4\pi \rho AB^2, \quad (19)$$

$$\dot{\psi}' - \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} - 2\frac{\dot{C}}{C} \right) \psi' = 0, \quad (20)$$

where the first equation is for $\alpha = 0$ and the second is for $\alpha = 1$. Here the dot and the prime represent the partial derivatives with respect to t and r respectively. Integrating (19) we obtain

$$\psi' = \frac{2sAB}{C^2}, \quad (21)$$

where

$$s(r) = 2\pi \int_0^r \rho BC^2 dr \quad (22)$$

is the total charge distributed per unit length of the cylinder. Equation (21) is in conformity with the conservation of charge and satisfies Eq. (20).

C. The Field Equations

We now find the field equations for this distribution of fluid. The Einstein field equations for the interior metric can be written as

$$G_{\alpha\beta}^- = 8\pi(T_{\alpha\beta}^- + T_{\alpha\beta}^{(em)-}) \quad (23)$$

where $G_{\alpha\beta}^-$ is the Einstein tensor for the interior metric. There are five non-zero components of (23) for the metric (1) with energy-momentum tensor (2), which are

$$G_{00}^- = 8\pi(T_{00}^- + T_{00}^{(em)-})$$

i.e.

$$8\pi(\mu + \epsilon)A^2 + \frac{4s^2A^2}{C^4} = \frac{\dot{C}}{C} \left(2\frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right) + \left(\frac{A}{B} \right)^2 \left(-2\frac{C''}{C} + \frac{C'}{C} \left(2\frac{B'}{B} - \frac{C'}{C} \right) \right). \quad (24)$$

Similarly,

$$G_{01}^- = 8\pi(T_{01}^- + T_{01}^{(em)-}),$$

which yields

$$8\pi(q + \epsilon)AB = 2 \left(\frac{\dot{C}'}{C} - \frac{\dot{B}C'}{BC} - \frac{\dot{C}A'}{CA} \right). \quad (25)$$

The remaining equations are

$$\begin{aligned}
G_{11}^- &= 8\pi(T_{11}^- + T_{11}^{(em)-}) \\
&= 8\pi\left(P_r + \epsilon - \frac{4}{\sqrt{3}}\eta\sigma\right)B^2 - \frac{4s^2B^2}{C^4} = 8\pi(P_{r_{eff}} + \epsilon)B^2 - \frac{4s^2B^2}{C^4} \\
&= -\left(\frac{B}{A}\right)^2\left(2\frac{\ddot{C}}{C} + \left(\frac{\dot{C}}{C}\right)^2 - 2\frac{\dot{A}\dot{C}}{AC}\right) + \left(\frac{C'}{C}\right)^2 + 2\frac{A'C'}{AC},
\end{aligned} \tag{26}$$

where the effective radial pressure is defined as

$$P_{r_{eff}} = P_r - \frac{4}{\sqrt{3}}\eta\sigma$$

and

$$\begin{aligned}
G_{22}^- &= 8\pi(T_{22}^- + T_{22}^{(em)-}) = 8\pi\left(P_\perp + \frac{2}{\sqrt{3}}\eta\sigma\right)C^2 + \frac{4s^2}{C^2} = 8\pi P_{\perp_{eff}}C^2 + \frac{4s^2}{C^2} \\
&= -\left(\frac{C}{A}\right)^2\left(\frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} - \frac{\dot{A}}{A}\left(\frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) + \frac{\dot{B}\dot{C}}{BC}\right) \\
&\quad + \left(\frac{C}{B}\right)^2\left(\frac{A''}{A} + \frac{C''}{C} - \frac{A'}{A}\left(\frac{B'}{B} - \frac{C'}{C}\right) - \frac{B'C'}{BC}\right),
\end{aligned} \tag{27}$$

with the effective tangential pressure as

$$P_{\perp_{eff}} = P_\perp + \frac{2}{\sqrt{3}}\eta\sigma.$$

III. EXTERIOR METRIC AND THE JUNCTION CONDITIONS

Exterior to the hypersurface Σ in the 4D manifold V^+ , we consider Vaidya's metric [8] in presence of charge in the retarded time coordinate as considered by Chao-Guang [43], but with a signature flip. The introduction of the retarded time coordinate removes the singularities of the original line element. Let $M(u)$ and $Q(u)$ be the mass and charge of the fluid respectively inside the hypersurface Σ , where u is the retarded time coordinate. Then the exterior field in this cylindrically symmetric spacetime can be defined as

$$ds_+^2 = -\left(\frac{-2M(u)}{R} + \frac{Q^2(u)}{R_\Sigma^2}\right)du^2 - 2dRdu + R^2(d\theta^2 + dz^2). \tag{28}$$

The intrinsic metric for the hypersurface Σ which enables a description in comoving coordinates of the interior spacetime, is given by [10]

$$(ds^2)_\Sigma = -d\tau^2 + R_\Sigma^2(\tau)(d\theta^2 + dz^2), \tag{29}$$

where $(\)_\Sigma$ means the value of $(\)$ on Σ and $\xi^i \equiv (\tau, \theta, z)$ represents the coordinates on Σ , i.e.

$$(ds^2)_\Sigma = g_{ij}d\xi^i d\xi^j.$$

To match the interior and the exterior space-time, we follow the prescription of Darmois and Israel [44] which demands:

- The first fundamental form must be continuous over the hypersurface Σ i.e., the continuity of the metrics as V^\pm approaches Σ :

$$(ds^2)_\Sigma = (ds_-^2)_\Sigma = (ds_+^2)_\Sigma. \tag{30}$$

- The continuity of the second fundamental form. This gives the continuity of the extrinsic curvature K_{ij} at the hypersurface Σ :

$$[K_{ij}] = K_{ij}^+ - K_{ij}^- = 0. \tag{31}$$

According to Eisenhart [45], the extrinsic curvature of Σ is given by

$$K_{ij}^{\pm} = -n_{\sigma}^{\pm} \left(\frac{\partial^2 \chi_{\pm}^{\sigma}}{\partial \xi^i \partial \xi^j} + \Gamma_{\mu\nu}^{\sigma} \frac{\partial \chi_{\pm}^{\mu}}{\partial \xi^i} \frac{\partial \chi_{\pm}^{\nu}}{\partial \xi^j} \right), \quad (\sigma, \mu, \nu = 0, 1, 2, 3). \quad (32)$$

where n_{σ}^{\pm} are the outward unit normal vectors to the hypersurface Σ , $\chi^{\pm\mu}$ are the coordinates of V^{\pm} .

On the $r = \text{constant}$ hypersurface, we have $dr = 0$. Using this condition in (1) and comparing with (29) keeping in mind the junction condition (30), we get,

$$\frac{dt}{d\tau} = A(t, r_{\Sigma})^{-1}, \quad (33)$$

$$R_{\Sigma}(\tau) = C(t, r_{\Sigma}).$$

We may also write the exterior metric (28) as,

$$(ds_+^2)_{\Sigma} = - \left[\left(\frac{-2M(u)}{R_{\Sigma}} + \frac{Q^2(u)}{R_{\Sigma}^2} \right) + \frac{2dR_{\Sigma}}{du} \right] du^2 + R_{\Sigma}^2 (d\theta^2 + dz^2). \quad (34)$$

Now, using the junction condition (30) and matching with the metric on the hypersurface Σ , we get

$$\frac{du}{d\tau} = \left[\frac{-2M(u)}{R_{\Sigma}} + \frac{Q^2(u)}{R_{\Sigma}^2} + \frac{2dR_{\Sigma}}{du} \right]^{-1/2}. \quad (35)$$

In order that we can apply the junction conditions, we require that Σ has the same parametrisation whether it is considered as embedded in V^+ or in V^- . In the coordinates of the interior spacetime V^- , the bounding surface Σ will have the equation

$$f(t, r) = r - r_{\Sigma} = 0, \quad (36)$$

where r_{Σ} is a constant.

Since the vector $\partial f / \partial \chi_{-}^{\alpha}$ is orthogonal to Σ , so the unit normal vector to Σ in the χ_{-}^{α} coordinate system is,

$$n_{\alpha}^{-} = 0, B(t, r_{\Sigma}), 0, 0. \quad (37)$$

In the coordinate system of V^+ , the equation for the surface Σ may be written as,

$$f(u, R) = R - R_{\Sigma}(u) = 0. \quad (38)$$

The vector $\partial f / \partial \chi_{+}^{\alpha}$, orthogonal to the hypersurface Σ is therefore given by,

$$\frac{\partial f}{\partial \chi_{+}^{\alpha}} = \left(-\frac{dR_{\Sigma}}{du}, 1, 0, 0 \right). \quad (39)$$

So the unit normal to Σ in the V^+ coordinate system is,

$$n_{\alpha}^{+} = \left(\frac{-2M(u)}{R_{\Sigma}} + \frac{Q^2(u)}{R_{\Sigma}^2} + \frac{2dR_{\Sigma}}{du} \right)^{-1/2} \left(-\frac{dR_{\Sigma}}{du}, 1, 0, 0 \right). \quad (40)$$

The extrinsic curvature for the hypersurface Σ in the V^+ coordinates as calculated by using (32) is given by

$$K_{00}^{-} = - \left(\frac{A'}{AB} \right)_{\Sigma}, \quad (41)$$

$$K_{22}^{-} = K_{33}^{-} = \left(\frac{CC'}{B} \right)_{\Sigma}, \quad (42)$$

$$K_{00}^{+} = \left[\frac{d^2 u}{d\tau^2} \left(\frac{du}{d\tau} \right)^{-1} - \left(\frac{M}{R^2} - \frac{Q^2}{R^3} \right) \left(\frac{du}{d\tau} \right) \right]_{\Sigma}. \quad (43)$$

$$K_{22}^{+} = K_{33}^{+} = \left[R \frac{dR}{d\tau} + \left(\frac{Q^2}{R} - 2M \right) \frac{du}{d\tau} \right]_{\Sigma}. \quad (44)$$

On account of the continuity of the second fundamental form given by (31), we obtain the following relations on matching (41) to (43) and (42) to (44)

$$\left[\frac{d^2 u}{d\tau^2} \left(\frac{du}{d\tau} \right)^{-1} - \left(\frac{M}{R^2} - \frac{Q^2}{R^3} \right) \left(\frac{du}{d\tau} \right) \right]_{\Sigma} = - \left(\frac{A'}{AB} \right)_{\Sigma}, \quad (45)$$

$$\left[R \frac{dR}{d\tau} + \left(\frac{Q^2}{R} - 2M \right) \frac{du}{d\tau} \right]_{\Sigma} = \left(\frac{CC'}{B} \right)_{\Sigma}. \quad (46)$$

IV. RESULTS

We now use the relations obtained above and simplify them to find useful results. From (35) we have by rearranging,

$$\left(\frac{du}{d\tau} \right) \left(\frac{Q^2}{R_{\Sigma}} - 2M \right) = R_{\Sigma} \left(\frac{du}{d\tau} \right)^{-1} - 2R_{\Sigma} \left(\frac{dR_{\Sigma}}{d\tau} \right). \quad (47)$$

Putting this value in (46) and using (33), we have

$$\left(\frac{du}{d\tau} \right)^{-1} = \left(\frac{\dot{C}}{A} + \frac{C'}{B} \right). \quad (48)$$

Again using (33) and squaring (48) we obtain the total energy entrapped inside the surface Σ as follows:

$$M = \frac{C}{2} \left(\left(\frac{\dot{C}}{A} \right)^2 - \left(\frac{C'}{B} \right)^2 \right) + \frac{Q^2}{2C}. \quad (49)$$

Taking the interior and exterior charge to be the same on the hypersurface Σ (i.e. $Q = s$) and using (49) and (14), we obtain

$$E' = \frac{l}{8} + M, \quad (50)$$

which indicates that the difference between the two masses is equal to $l/8$, as obtained in [38] and [39], which is a consequence of the least unsatisfactory definition of C-energy due to Thorne [41].

Using the expressions (8), (9) and (12) we can reconstruct (25) as follows:

$$4\pi(q + \epsilon) = \frac{1}{B} \left(\frac{1}{3}(\Theta - \sqrt{3}\sigma)' - \sqrt{3}\sigma \frac{C'}{C} \right). \quad (51)$$

Differentiating (48) with respect to τ and substituting in (45), we obtain the following expression with the help of (48) and (33),

$$\frac{C}{A^2} \left(\frac{\ddot{C}}{C} - \frac{\dot{C}\dot{A}}{CA} \right) + \frac{C}{AB} \left(\frac{\dot{C}'}{C} - \frac{\dot{B}C'}{BC} \right) - \frac{1}{AB} \left(\frac{\dot{C}A'}{A} + \frac{C'A'}{B} \right) = \frac{1}{C^2} \left(\frac{Q^2}{C} - M \right). \quad (52)$$

Using (49), (25) and (26) in (52) and rearranging terms, we arrive at the result

$$q = \left(P_r - \frac{4}{\sqrt{3}}\eta\sigma \right) - \frac{3s^2}{8\pi C^4}, \quad (53)$$

on account of the fact that $Q = s$ on the hypersurface Σ . This equation gives the relation between the heat flux, radial pressure, shear viscosity and the charge per unit length of the cylinder, over the hypersurface Σ . The result shows that for an uncharged radiating fluid without any shear viscosity, the radial pressure equals the heat flux all over the boundary of the collapsing cylinder. Equations (51) and (53) are generalizations over the results obtained earlier in [38] and [39].

The total luminosity of the collapsing matter visible to an observer at rest at infinity is

$$L_{\infty} = - \left(\frac{dM}{du} \right)_{\Sigma} = - \left(\frac{dM}{dt} \frac{dt}{d\tau} \left(\frac{du}{d\tau} \right)^{-1} \right)_{\Sigma}. \quad (54)$$

Differentiating (49) with respect to t and using (33), (48), (25) and (26), we obtain

$$L_\infty = 4\pi C^2 \left(\frac{\dot{C}}{A} \left(\left(P_r + \epsilon - \frac{4}{\sqrt{3}}\eta\sigma \right) - \frac{3s^2}{8\pi C^4} \right) + \frac{C'}{B}(q + \epsilon) \right) \left(\frac{\dot{C}}{A} + \frac{C'}{B} \right), \quad (55)$$

which, in view of (53) leads us to the expression

$$L_\infty = 4\pi \left[C^2(q + \epsilon) \left(\frac{\dot{C}}{A} + \frac{C'}{B} \right)^2 \right]_\Sigma. \quad (56)$$

Thus the total luminosity of the collapsing matter as visible to a distant observer, depends on the energy flux associated with the collapse. Since Chiba [31] has demonstrated the absence of horizons in the cylindrical system, we expect the end state in this case to be a naked singularity. Therefore the collapse will terminate at the time when

$$\left(\frac{\dot{C}}{A} + \frac{C'}{B} \right) = 0.$$

A. Dynamical Equations for the Collapse

The dynamical equations for non-adiabatic charged anisotropic fluid with shear viscosity undergoing cylindrical collapse can be obtained from the Bianchi identities $(T^{\alpha\beta} + T^{(em)\alpha\beta})_{;\beta} = 0$ for energy-momentum conservation. Using (2), (6), (7) and (15), we have

$$\begin{aligned} \left(T^{\alpha\beta} + T^{(em)\alpha\beta} \right)_{;\beta} V_\alpha &= -\frac{1}{A}(\dot{\mu} + \dot{\epsilon}) - \frac{\dot{B}}{AB} \left(\mu + P_r + 2\epsilon - \frac{4}{\sqrt{3}}\eta\sigma \right) - \frac{2\dot{C}}{AC} \left(\mu + P_\perp + \epsilon + \frac{2}{\sqrt{3}}\eta\sigma \right) \\ &\quad - \frac{2(q + \epsilon)}{B} \left(\frac{A'}{A} + \frac{C'}{C} \right) - \frac{1}{B}(q' + \epsilon') = 0 \end{aligned} \quad (57)$$

and

$$\begin{aligned} \left(T^{\alpha\beta} + T^{(em)\alpha\beta} \right)_{;\beta} \chi_a &= \frac{1}{B} \left(P_r + \epsilon - \frac{4}{\sqrt{3}}\eta\sigma \right)' + \frac{A'}{AB} \left(\mu + P_r + 2\epsilon - \frac{4}{\sqrt{3}}\eta\sigma \right) + \frac{2C'}{BC} \left(P_r - P_\perp + \epsilon - 2\sqrt{3}\eta\sigma \right) \\ &\quad + \frac{1}{A}(\dot{q} + \dot{\epsilon}) + \frac{2(q + \epsilon)}{A} \left(\frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right) - \frac{ss'}{\pi BC^4} = 0. \end{aligned} \quad (58)$$

To discuss the dynamics of the collapsing system, it is customary to introduce the proper time derivative

$$D_T = \frac{1}{A} \frac{\partial}{\partial t}, \quad (59)$$

and the proper radial derivative D_R constructed from the circumference radius of a cylinder inside Σ

$$D_R = \frac{1}{R'} \frac{\partial}{\partial r}, \quad (60)$$

where

$$R = C. \quad (61)$$

The fluid velocity for the corresponding collapse is given by

$$U = D_T(R) = \frac{\dot{C}}{A}, \quad (62)$$

which must be negative to ensure collapse to occur. Defining new variable $\varepsilon = \frac{C'}{B}$ (note that ϵ and ε are different quantities) and using (14), we have

$$\varepsilon = \left[U^2 + \frac{s^2}{C^2} - \frac{2M}{C} \right]^{1/2}. \quad (63)$$

Consequently, Eq. (51) can be re-written as follows:

$$4\pi(q + \epsilon) = \varepsilon \left[\frac{1}{3} D_R(\Theta - \sqrt{3}\sigma) - \sqrt{3} \frac{\sigma}{R} \right]. \quad (64)$$

The time rate of variation of the total energy inside the collapsing cylinder is given by

$$D_T E' = -4\pi R^2 \left[\left(P_r + \epsilon - \frac{4}{\sqrt{3}} \eta \sigma - \frac{1}{32\pi R^2} \right) U + \varepsilon(q + \epsilon) \right] + \frac{3s^2 U}{2R^2}. \quad (65)$$

In the case of collapse, since ($U < 0$), the coefficient of U inside the square brackets, will increase the C-energy of the cylinder if $P_r + \epsilon - \frac{4}{\sqrt{3}} \eta \sigma > \frac{1}{32\pi R^2}$, i.e. the effective radial pressure is greater than a certain value. The work done by the effective radial pressure leads to the increase of C-energy. The second term in the square brackets, due to the overall negative sign, describes the outflow of energy in the form of heat flux and radiation during the collapse. Since the collapsing cylinder contains the same species of the charges, the last term will decrease the energy of the system as $\frac{3s^2}{2R^2}$ plays the role of Coulomb force of repulsion and $U < 0$.

The variation of energy between the adjacent coaxial cylinders inside the fluid is given by the expression

$$D_R E' = 4\pi R^2 \left(\mu + \epsilon + \frac{U}{\varepsilon}(q + \epsilon) \right) + \frac{l}{8} + \frac{s}{R} D_R s + \frac{3s^2}{2R^2}. \quad (66)$$

The first term on the right hand side gives the contribution of the energy density of the element of fluid inside a cylindrical shell, along with heat flux and radiation. Since $U < 0$, the factor $\frac{U}{\varepsilon}(q + \epsilon)$ decreases the energy of the system during the collapse of the cylinder. In the remaining terms, the constant $l/8$ comes from the definition of C-energy and the other term is the electromagnetic contribution. The C-energy of the cylinder at a given instant of time is then obtained by integrating (66) from the axis to the periphery of the cylinder. The acceleration of the collapsing matter inside the hypersurface Σ is obtained using (14), (26), (62) and (63)

$$D_T U = -\frac{1}{R^2} \left(E' - \frac{l}{8} \right) - 4\pi R \left(P_r + \epsilon - \frac{4}{\sqrt{3}} \eta \sigma \right) + \frac{\varepsilon A'}{AB} + \frac{5s^2}{2R^3}. \quad (67)$$

Substituting for $\frac{A'}{A}$ from Eq.(67) into Eq.(58), we obtain the equivalent of Newton's second law of motion for the collapsing matter in the form

$$\begin{aligned} \left(\mu + P_r + 2\epsilon - \frac{4}{\sqrt{3}} \eta \sigma \right) D_T U = & - \left(\mu + P_r + 2\epsilon - \frac{4}{\sqrt{3}} \eta \sigma \right) \left[\frac{1}{R^2} \left(E' - \frac{l}{8} \right) + 4\pi \left(P_r + \epsilon - \frac{4}{\sqrt{3}} \eta \sigma \right) R - \frac{5s^2}{2R^3} \right] \\ & - \varepsilon \left[D_T(q + \epsilon) + \frac{4(q + \epsilon)U}{R} + 2(q + \epsilon) \frac{1}{A} \left(\frac{\dot{B}}{B} - \frac{\dot{C}}{C} \right) \right] \\ & - \varepsilon^2 \left[D_R \left(P_r + \epsilon - \frac{4}{\sqrt{3}} \eta \sigma \right) + 2 \left(P_r - P_\perp + \epsilon - 2\sqrt{3} \eta \sigma \right) \frac{1}{R} - \frac{s}{\pi R^4} D_R s \right], \end{aligned} \quad (68)$$

which can be simplified as follows:

$$\begin{aligned} (\mu + P_{r_{eff}} + 2\epsilon) D_T U = & - (\mu + P_{r_{eff}} + 2\epsilon) \left[\frac{1}{R^2} \left(E' - \frac{l}{8} \right) + 4\pi (P_{r_{eff}} + \epsilon) R - \frac{5s^2}{2R^3} \right] \\ & - \varepsilon \left[D_T(q + \epsilon) + \frac{4(q + \epsilon)U}{R} + \frac{2(q + \epsilon)}{A} \left(\frac{\dot{B}}{B} - \frac{\dot{C}}{C} \right) \right] \\ & - \varepsilon^2 \left[D_R(P_{r_{eff}} + \epsilon) + 2(P_{r_{eff}} - P_{\perp_{eff}} + \epsilon) \frac{1}{R} - \frac{s}{\pi R^4} D_R s \right]. \end{aligned}$$

Here we assume that in general $\frac{1}{A} \left(\frac{\dot{B}}{B} - \frac{\dot{C}}{C} \right) \neq 0$. The left hand side of (68) represents force. The factor $(\mu + P_r + 2\epsilon - \frac{4}{\sqrt{3}} \eta \sigma)$ represents the inertial mass density, which gives the effect of dissipation but there is no contribution of the electric charge, nor of heat flux. The remaining term on the left hand side is acceleration. Thus, we can say that the dynamical system will evolve radially outward or inward according as $D_T U < 0$ or $D_T U > 0$. The terms with a

negative contribution in (68), favors the collapse while the other contribution prevents the collapse. If both of these terms cancel each other, then a condition of hydrostatic equilibrium will be encountered.

The first term on the right hand side represents the gravitational force. The factor within the first square brackets shows the effects of specific length, effective radial pressure and the electric charge on the term $(\mu + P_r + 2\epsilon - \frac{4}{\sqrt{3}}\eta\sigma)$ representing the active gravitational mass. The second term represents the contribution due to radiation and heat flux, which will leave the system (if there is an overall negative sign) through the outward radially directed streamlines. Thus it is in the same direction of pressure and would prevent the collapse. The third term has three main contributions: the first is the effective pressure gradient which is always negative, thereby preventing the collapse. The second is the local anisotropy of the fluid which will be negative for $P_{reff} < P_{\perp eff}$, in which case it will decrease the rate of collapse. The third is the electromagnetic field term. According to [21], the third term contributes negatively if $\frac{s}{R} > D_{Rs}$. Under these conditions, the term in the third square brackets, with negative sign, contributes positively by reducing the attractive nature of the force appearing on the left hand side of this equation and hence this term will prevent the gravitational collapse.

B. Solution of the Field Equations

In their work, Di Prisco *et al.* [37] have derived the solutions for the shearfree and isotropic case of cylindrical collapse. Sharif and Abbas [46] have found analytical solutions for charged perfect fluid cylindrical gravitational collapse. The general solution in presence of dissipation, using the method of separation of variables, should be of the following form:

$$\begin{aligned} A(t, r) &= A_0(r), \\ B(t, r) &= B_0(r)f_1(t), \\ C(t, r) &= C_0(r)f_2(t), \end{aligned} \quad (69)$$

where $A_0(r)$, $B_0(r)$ and $C_0(r)$ are solutions of a static fluid having μ_0 as the energy density and p_{r0} and $p_{\perp 0}$ as the radial and tangential pressure. These solutions for such a general case, are highly complicated and hence we concentrate on a simplified choice for the facility of calculation. For that we extend the solutions assumed in [18] for the spherically symmetric distribution to the case of dissipative cylindrical collapse with the choice of $f_1(t) = 1$ and $f_2(t) = f(t)$, in Eq. (69) above, i.e.

$$\begin{aligned} A(t, r) &= A_0(r), \\ B(t, r) &= B_0(r), \\ C(t, r) &= C_0(r)f(t). \end{aligned} \quad (70)$$

On account of equation (33) in the junction condition, the function $C(t, r_\Sigma)$ represents the luminosity radius of the collapsing matter as seen by a distant observer. The expression (12) for the shear scalar becomes

$$\sigma = -\frac{1}{\sqrt{3}A_0} \frac{\dot{f}}{f}. \quad (71)$$

The field equations (24) to (27) are reduced to

$$8\pi(\mu + \epsilon) = 8\pi\mu_0 + \frac{\dot{f}^2}{A_0^2 f^2} - \frac{4s^2}{C_0^4 f^4}, \quad (72)$$

$$8\pi(q + \epsilon) = \frac{2\dot{f}}{A_0 B_0 f} \left(\frac{C'_0}{C_0} - \frac{A'_0}{A_0} \right) \quad (73)$$

$$8\pi(P_r + \epsilon) = 8\pi P_{r0} - \frac{1}{A_0^2} \left(\frac{2\ddot{f}}{f} + \left(\frac{\dot{f}}{f} \right)^2 \right) - \frac{32\pi\eta\dot{f}}{3A_0 f} + \frac{4s^2}{C_0^4 f^4}, \quad (74)$$

$$8\pi P_\perp = 8\pi P_{\perp 0} - \frac{\ddot{f}}{A_0^2 f} + \frac{16\pi\eta\dot{f}}{3A_0 f} - \frac{4s^2}{C_0^4 f^4} \quad (75)$$

where

$$\mu_0 = \frac{1}{B_0^2} \left[-\frac{2C_0''}{C_0} + \frac{C_0'}{C_0} \left(\frac{2B_0'}{B_0} - \frac{C_0'}{C_0} \right) \right], \quad (76)$$

$$8\pi P_{r0} = \frac{1}{B_0^2} \left(\left(\frac{C_0'}{C_0} \right)^2 - \frac{2A_0' C_0'}{A_0 C_0} \right) \quad (77)$$

and

$$8\pi P_{\perp 0} = \frac{1}{B_0^2} \left(\frac{A_0''}{A_0} + \frac{C_0''}{C_0} - \frac{A_0'}{B_0'} \left(\frac{B_0'}{B_0} - \frac{C_0'}{C_0} \right) - \frac{B_0' C_0'}{B_0 C_0} \right). \quad (78)$$

Equations (72) to (75) represents the static anisotropic fluid configuration in the limit $f(t) \rightarrow 1$. Substituting (71), (73) and (74) into (53) and assuming that $P_{r0} = 0$, we obtain the following differential equation:

$$2f\ddot{f} + \dot{f}^2 + af\dot{f} - \frac{b}{f^2} = 0, \quad (79)$$

where a and b are functions of r , i.e.

$$a = \frac{2A_0}{B_0} \left(\frac{C_0'}{C_0} - \frac{A_0'}{A_0} \right) \quad (80)$$

and

$$b = \frac{A_0^2 s^2}{C_0^4}. \quad (81)$$

Equation (79) can be solved numerically assuming suitable boundary conditions. Although the first three terms on the left hand side are similar to Eq. (56) in [18], but this case is clearly distinct due to the difference in geometry and the nature of collapsing matter. Thus the parameters a and b are different in their content. The parameter b in our case, depends on the charge enclosed inside the cylinder and hence may be positive, negative or zero.

The solution is sought under the following boundary conditions:

1. Initially the system represents the static configuration at $t \rightarrow -\infty$, when $\dot{f}(t) \rightarrow 0$ and $f(t) \rightarrow 1$.
2. Further, if possible, as $t \rightarrow 0$, $f(t) \rightarrow 0$. This means that the luminosity radius $C(t, r_\Sigma)$ has the value $C_0(r_\Sigma)$ at the beginning of the collapse and vanishes as the final fate is approached.

Using the second boundary condition in (79), we find that $\dot{f}(t)$ becomes infinite as $t \rightarrow 0$. Thus the luminosity will remain fairly constant during the collapse and will suddenly go to zero value at the end. Moreover, applying the initial condition we find that the second order derivative of $f(t)$ i.e. $\ddot{f}(t)$ depends on the nature of the charge enclosed inside the cylinder at $t \rightarrow -\infty$. Therefore we can divide our subsequent analysis into two parts:

- **When the fluid is uncharged:** In this case the last term on the left hand side of (79) vanishes irrespective of the instant of time. Thus (79) is now reduced to the form $2f\ddot{f} + \dot{f}^2 + af\dot{f} = 0$. It is a second order, reducible differential equation of the Liouville type and can be solved using Maple or Mathematica program. The solution is a decreasing function of time, with the rate of decrease being dependent on the parameter a and is given by $f(t) = \left(\frac{3C_1}{2} - \frac{3C_2}{a} e^{-at/2} \right)^{2/3}$, where C_1 and C_2 are arbitrary constants. Expanding binomially, neglecting the higher order terms and choosing suitable values for C_1 and C_2 , we find that $f(t) \sim 1 - \exp(-at/2)$.
- **When the fluid is charged:** In the case of a charged fluid, the parameter b is non-zero, and the solution is not straightforward. If this charge is negative, then the second derivative will be negative at $t \rightarrow -\infty$ and hence $f(t)$ will be a decreasing function of time as $t \rightarrow 0$. When the charge inside the cylinder is positive, $f(t)$ will increase as $t \rightarrow 0$. To achieve a numerical solution, we modify our trial solution in the form $A(t, r) = A_0(r)$, $B(t, r) = B_0(r)$ and $C(t, r) = A_0(r)f(t)$. This corresponds to the case for which $(q + \epsilon) = 0$, although q and ϵ are separately non-zero. This feature arises due to the special type of geometry which is involved in this case and is not a general feature. The evolution equation is now: $2f\ddot{f} + \dot{f}^2 - \frac{b}{f^2} = 0$, which can be solved numerically by assuming sample values for b , but for that we have to discard the second boundary condition, i.e., we now have at $t \rightarrow 0$, $f(t) \neq 0$.

Sample plots are shown in the following figures.

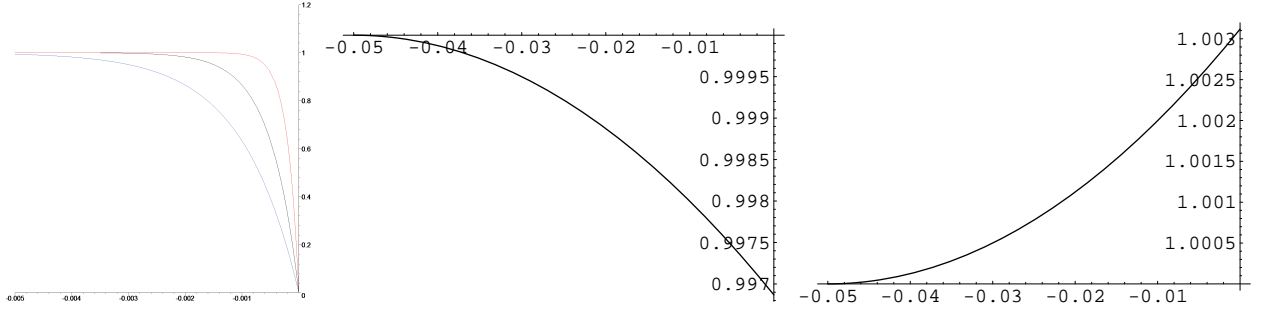


FIG. 1: Diagram showing the plot of $f(t)$ vs t for the uncharged case with increasing values of a in the left figure, the charged case with $b < 0$ in the middle figure and with $b > 0$ in the right figure. The magnitude of b is the same in the last two figures.

V. SUMMARY

We have formulated, by the help of General Relativity, the field equations for the case of dissipative cylindrical collapse in presence of heat flow, free-streaming radiation, and shear viscosity and have obtained a few results. We have derived the relation between the expansion Θ , the shear σ and the energy flowing out of the cylinder in the form of heat flux q and free-streaming radiation. By employing the Darmois-Israel junction condition for the smooth matching of interior and exterior spacetimes at the boundary Σ , we have verified the relation between the specific energy of the cylinder due to the electromagnetic field and the mass of the collapsing matter. The total luminosity of the collapsing matter as visible to a distant observer, depends on the energy flux associated with the collapse. This energy flux over the hypersurface Σ bounding the cylinder, is dependent on the effective radial pressure and the charge per unit length of the cylinder. The dynamical equations for the collapse is derived from the Bianchi identities with the help of Misner-Sharp formalism for this non-adiabatic, anisotropic and dissipative fluid and the equation for the effective Newton's second law of motion is constructed. Finally, we have presented the solution to the field equations for the given matter distribution undergoing cylindrical collapse and have interpreted its significance from a physical point of view.

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